CH 39 – LOGARITHMS

❑ *MELANIE'S ALLOWANCE*

Melanie tells her father that *she* will pay for her entire college education all by herself *if* he will agree to the following plan: He gives her 2¢ on the first day of the month, 4¢ on the second day of the month, 8¢ on the third day of the month, 16¢ on the fourth day of the month, and so on till the 30th of the month. After that month, no more money. Dad (who was a philosophy major) thinks this is a great money-saving idea for him and accepts Melanie's proposal.

Day	# Pennies
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
⋮	⋮
30	1,073,741,824 (almost $11 million)

In the chart we have calculated Melanie's earnings for each of the first 10 days of the month; then we cut to the chase and calculated the amount for the 30th day. Take your calculator and verify each of the 11 amounts of money that are in the second column of the table. You should start at the 10th day, and then double the 1024¢ to get 2048¢ for the 11th day, and keep doubling until you get to the 30th day.

Now let's come up with a direct formula that computes the money earned from the day of the month, <u>without having to know all the previous days' amounts</u>. Notice that each amount of money is simply 2 raised to the power of the day. For example, consider the 9th day. If we raise 2 to the 9th power (use the exponent button on your calculator), we get 512. That is, $2^9 = 512$. Now calculate 2^{30} and you should get 1,073,741,824.

Homework

1. a. How many pennies will Melanie earn on the 15th day of the month?

   b. How many pennies will Melanie earn the 31st of the month if Dad agreed to extend the plan that far?

   c. On which day of the month did Melanie earn half of what she earned on the 30th day of the month?

2. Now for a little practice in the concept to be covered in this chapter. I'll give you a penny amount, and you tell me which day of the month Melanie earned that amount of money.

   a. 512¢ b. 4096¢ c. 1,048,576¢ d. 33,554,432¢

3. Similar question, but a little more abstract: I'll give you a number, and you tell me the exponent that **2** would have to be raised to, in order to get the number I gave you. For example, if I give you the number 2048, then you say "11" because $2^{\boxed{11}} = 2048$.

   a. 2 b. 256 c. 64 d. 1
   e. 8192 f. 131,072 g. 524,288 h. 1/2

4. Another question like #3, but now when I give you the number, you tell me the exponent that **10** would have to be raised to, in order to get the number I gave you. For example, if I give you the number 1,000, then you say "3" because $10^{\boxed{3}} = 1000$.

   a. 100 b. 10,000 c. 10 d. 1
   e. 100,000 f. 1,000,000 g. 1 billion h. 1 googol

❑ THE MEANING OF A LOG

A ***log*** (short for ***logarithm***) is an exponent. It's the exponent that one number (called the base) must be raised to, in order to get a specified number. This definition is so far off in the clouds that we need to get to an example right now!

> *logarithm*
> from the Greek:
> *logos* = reason, plan
> *arithmos* = number

For our first example, to calculate

$\log_{10}(1000)$ [read: "log, base 10, of 1000" or "log of 1000, base 10]

we ask ourselves, "10 raised to what power equals 1000?" In other words, 10 to the "what" equals 1000? The answer is 3, since $10^3 = 1000$. Therefore,

$\log_{10}(1000) = \mathbf{3}$ [log, base 10, of 1000 is 3.]

For a second example, let's analyze $\log_2 32$, which asks us, "2 raised to the "what" equals 32?" Well, 2 to the 5th power equals 32, and so

$\log_2(32) = \mathbf{5}$ [log, base 2, of 32 is 5.]

Our third example will describe a log a little differently: If you can fill in the box in the equation $4^{\square} = 16$, then you have found the "log, base 4, of 16," which is 2. That is,

$\log_4(16) = \mathbf{2}$

Notation: A log is a function, so notation like $\log_2(128)$ certainly makes sense, just like when we use parentheses in function notation: the classic $f(x)$. But if it's clear what we're taking the log of, we don't really need the parentheses; so, for example, $\log_2(128)$ is simply written $\log_2 128$. [Although in computer programming, the parentheses <u>are</u> required.]

Summary: $\log_{10} 1000 = 3$ because $10^{\mathbf{3}} = 1000$

$\log_2 32 = 5$ because $2^{\mathbf{5}} = 32$

$\log_4 16 = 2$ because $4^{\mathbf{2}} = 16$

This is really abstract, isn't it? Let's get right to some homework.

Homework

5. To find $\log_5 25$, which is read "log, base 5, of 25," ask yourself "5 raised to what power equals 25?" $5^{\square} = 25$

6. To find $\log_2 8$, which is read "log, base 2, of 8," ask yourself "2 raised to what power equals 8?" $2^{\square} = 8$

7. To find $\log_9 9$, which is read "log, base 9, of 9," ask yourself "9 raised to what power equals 9?" $9^{\square} = 9$

8. To find $\log_{17} 1$, which is read "log, base 17, of 1," ask yourself "17 raised to what power equals 1?" $17^{\square} = 1$

9. To find $\log_{100} 10$, which is read "log, base 100, of 10," ask yourself "100 raised to what power equals 10?" $100^{\square} = 10$

10. To find $\log_6\left(\frac{1}{6}\right)$, which is read "log, base 6, of 1/6," ask yourself "6 raised to what power equals $\frac{1}{6}$?" $6^{\square} = \frac{1}{6}$

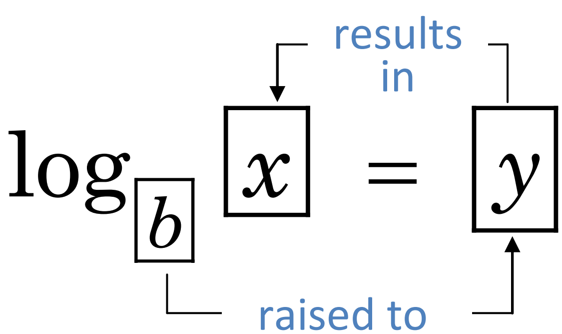
❑ THE OFFICIAL DEFINITION OF LOG

$$\log_b x = y \quad \text{means} \quad b^y = x$$

The notation "$\log_b x$" is read either

"log, base *b,* of *x*" or "log of *x*, base *b*"

Here's another way to visualize the meaning of logarithm:



EXAMPLE 1:

A. $\log_{10} 10{,}000 = \mathbf{4}$ Why? Because $10^4 = 10{,}000$

B. $\log_e e^2 = \mathbf{2}$ because $e^2 = e^2$

[Here, e is the irrational number, from Chapter 37, whose value ≈ 2.718]

C. $\log_{17} 17 = \mathbf{1}$ because $17^1 = 17$

D. $\log_e 1 = \mathbf{0}$ because $e^0 = 1$

E. $\log_{25} 5 = \mathbf{\frac{1}{2}}$ because $25^{1/2} = \sqrt{25} = 5$

F. $\log_{64} 4 = \mathbf{\frac{1}{3}}$ because $64^{1/3} = \sqrt[3]{64} = 4$

G. $\log_8 4 = \mathbf{\frac{2}{3}}$ because $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

H. $\log_{13}\left(\frac{1}{13}\right) = \mathbf{-1}$ because $13^{-1} = \frac{1}{13}$

I. $\log_6\left(\frac{1}{36}\right) = \mathbf{-2}$ because $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

J. $\log_{49}\left(\frac{1}{7}\right) = \mathbf{-\frac{1}{2}}$ because $49^{-1/2} = \frac{1}{49^{1/2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

K. $\log_{1/2}\left(\frac{1}{8}\right) = \mathbf{3}$ because $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

<u>EXAMPLE 2:</u>

A. $\log_b b = \mathbf{1}$ since $b^1 = b$

B. $\log_b 1 = \mathbf{0}$ since $b^0 = 1$

C. $\log_b \frac{1}{b} = -1$ since $b^{-1} = \frac{1}{b}$

D. $\log_b \sqrt{b} = \frac{1}{2}$ since $b^{1/2} = \sqrt{b}$

E. $\log_b \sqrt[n]{b} = \frac{1}{n}$ since $b^{1/n} = \sqrt[n]{b}$

F. $\log_b b^n = \boldsymbol{n}$ since $b^n = b^n$

Homework

Find the value of each log:

11.	a. $\log_{10} 100$	b. $\log_5 125$	c. $\log_8 64$	d. $\log_2 64$
12.	a. $\log_e e^5$	b. $\log_b b^2$	c. $\log_{\sqrt{2}} \sqrt{2}$	d. $\log_L L$
13.	a. $\log_{10} 1$	b. $\log_e 1$	c. $\log_{\sqrt[5]{99}} 1$	d. $\log_b 1$
14.	a. $\log_{36} 6$	b. $\log_{49} 7$	c. $\log_{144} 12$	d. $\log_b \sqrt{b}$
15.	a. $\log_5 \left(\frac{1}{5}\right)$	b. $\log_e \left(\frac{1}{e}\right)$	c. $\log_{1/e} 1$	d. $\log_n \left(\frac{1}{n}\right)$
16.	a. $\log_Q Q^n$	b. $\log_x 1$	c. $\log_{2.3} 2.3$	d. $\log_9 81$
17.	a. $\log_8 2$	b. $\log_{64} 4$	c. $\log_{125} 5$	d. $\log_a \sqrt[3]{a}$

❑ *CALCULATING LOGS*

The homework problems above were designed so you could solve them by inspection (that is, with a little experimentation and insight). Some logs aren't easy to do that way. So now we present a longer, but more systematic, way of evaluating logs by solving certain exponential equations.

EXAMPLE 3: **Calculate: $\log_{27}9$**

Solution: Let's give our log expression a name — call it *y*. Now we can write an equation:

$$\log_{27}9 = y$$

The definition of log shows us how we can convert our *log* equation into an *exponential equation*:

$$27^y = 9$$

And now we solve for *y*. The previous chapter showed us how:

$$27^y = 9 \Rightarrow (3^3)^y = 3^2 \Rightarrow 3^{3y} = 3^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

But *y* was the name we gave to the original log problem. So we can conclude that

$$\boxed{\log_{27}9 = \frac{2}{3}}$$

To **check** our result, we can raise 27 to the $\frac{2}{3}$ power and make sure it comes out 9:

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9 \quad ✓$$

Homework

Find the value of each log:

18. $\log_{64}16$
19. $\log_{25}\left(\frac{1}{5}\right)$
20. $\log_{16}8$
21. $\log_{27}\left(\frac{1}{9}\right)$
22. $\log_{8}16$
23. $\log_{32}\left(\frac{1}{2}\right)$

24. $\log_{1/3}\left(\frac{1}{9}\right)$

25. $\log_{1/4} 64$

26. $\log_9 1$

27. $\log_\pi \pi^5$

28. $\log_\pi \sqrt{\pi}$

29. $\log_4\left(\frac{1}{8}\right)$

30. $\log_5\left(\frac{1}{25}\right)$

31. $\log_3\left(\frac{1}{\sqrt{3}}\right)$

32. $\log_4\left(\frac{1}{64}\right)$

33. $\log_2\left(\frac{1}{128}\right)$

34. $\log_{10} \sqrt{10}$

35. $\log_{10} \sqrt[3]{100}$

36. $\log_{10}\left(\frac{1}{100}\right)$

37. $\log_{10}\left(\frac{1}{\sqrt{10}}\right)$

38. $\log_{10}\left(\frac{1}{\sqrt{1000}}\right)$

❑ *THE pH SCALE FOR ACIDS AND BASES*

One use of **logs** is in the definition of the ***pH scale*** for acids and bases. The official definition of the **pH** of a substance is the negative **logarithm** (base 10) of the hydrogen-ion concentration of the substance. Acids (like lemonade) have a pH smaller than 7, while bases (like Drano, the drain cleaner) have a pH higher than 7. The pH of pure water is a neutral 7. The word *alkali* is another term for base.

We'll use the official chemistry notation for the hydrogen-ion concentration, $[H^+]$, which has the units of moles/liter. It is not necessary to understand any of the chemistry, or even what a mole is; indeed, the math takes care of everything. Devised by a biochemist while working on the brewing of beer, the **pH** of a substance is defined to be the negative logarithm (base 10) of the hydrogen ion concentration:

$$\text{pH} = -\log_{10}[H^+]$$

New Notation and Calculator Hints:

1. On a TI-30, to enter a number in scientific notation like 1.6×10^{-13}, first press 1.6, then press the "EE" button, and then press 13, and last the +/– key. Your display should then look something like 1.6 $^{-13}$ (the base of 10 is understood).

2. To find the "log, base 10, of 1000," $\log_{10} 1000$, enter 1000 into your calculator and then press the ***log*** button. You should, of course, get an answer of 3. On newer calculators, try pressing the *log* button first, followed by 1000. Using 10 as a base for logs is so "common" that it is officially referred to as the **common log**, and we dispense with writing the base of 10 — it's "understood":

$$\log_{10} x \text{ is written } \log x$$

Logs are also used in the definition of the *Richter Scale* for earthquakes, and for the *decibel scale* for measuring the loudness of sound.

3. A base of ***e*** is so important and occurs so "naturally" in the physical, the biological, and the business worlds that "log, base ***e***" also gets its own name and notation:

$$\log_e x \text{ is written } \ln x$$

You read "ln *x*" either as "el en *x*" or "el en of *x*" or "the **natural log** of *x*." Teachers will many times write it on the whiteboard in cursive:

$\ell n\, x$ (ℓ for **l**og, n for **n**atural)

EXAMPLE 4: **A sample of orange juice has a hydrogen-ion concentration of 2.9×10^{-4} moles/liter. Find the pH of the orange juice.**

Solution: According to the definition,

$$pH = -\log_{10}[H^+] = -\log(2.9\times10^{-4}) \approx -(-3.54) = 3.54$$

According to the text next to the lemonade stand above, this should mean that orange juice is an acid, as indeed it is (the sour taste is one clue). Thus, the pH of the sample of orange juice is

$$\boxed{3.54}$$

Homework

39. The hydrogen ion concentration of household ammonia is 1.26×10^{-12} moles/liter. Find the pH of the ammonia. Is it an acid or a base?

40. Pure water has a hydrogen ion concentration of 1.0×10^{-7} moles/liter. Prove that water has a neutral pH of 7.

41. Find the pH of each substance given its molarity:

    a. 1.3×10^{-2} moles/L
    b. 2.8×10^{-6} moles/L
    c. 0.3×10^{-10} moles/L
    d. 9.2×10^{-12} moles/L
    e. 5.9×10^{-7} moles/L
    f. 8.0×10^{-1} moles/L

❑ THE RICHTER SCALE FOR EARTHQUAKES

In 1935 Charles Richter, from Cal Tech in Pasadena, CA, devised a scale for earthquakes called the Richter scale (what a coincidence!). If E represents the energy (in joules) of the earthquake, then the Richter magnitude M is given by

$$\boxed{M = \frac{2}{3}\log\left(\frac{E}{2.5\times10^{4}}\right)}$$

EXAMPLE 5: **The energy release of the 1906 San Francisco earthquake was 1.253×10^{16} joules. Find the *Richter* magnitude of the earthquake.**

Solution: According to Richter's formula,

$$M = \frac{2}{3}\log\left(\frac{1.253\times10^{16}}{2.5\times10^{4}}\right)$$

$$= \frac{2}{3}\log\left(5.012\times10^{11}\right)$$

$$= \frac{2}{3}(11.700)$$

$$= \boxed{7.8}$$

Homework

42. An earthquake releases 1.75×10^{11} joules of energy. What is the Richter magnitude of the quake?

43. A more serious quake releases 100 times as much energy as the one in the previous problem. Find the Richter magnitude.

Practice Problems

44. a. $\log_{10} 1{,}000{,}000 =$ b. $\log_3 3^7 =$ c. $\log_e e =$

d. $\log_7 1 =$ e. $\log_{16} 4 =$ f. $\log_9\left(\frac{1}{9}\right) =$

g. $\log_5\left(\frac{1}{25}\right) =$ h. $\log_{36}\left(\frac{1}{6}\right) =$ i. $\log_2 512 =$

45. a. $\log_a a =$ b. $\log_c 1 =$ c. $\log_7 7^N =$

46. The hydrogen ion concentration of an unknown liquid is 3.4×10^{-11} moles/L. Find the pH of the liquid.

47. The energy release of an earthquake is 1.23×10^{17} joules. Use the formula $M = \frac{2}{3}\log\left(\frac{E}{2.5 \times 10^4}\right)$ to find the Richter magnitude of the quake.

48. What kind of music do they play in a lumber camp?

Solutions

1. a. 32,768 b. 2,147,483,648 c. I'm not gonna tell you that.

2. a. day 9 b. day 12 c. day 20 d. day 25

3. a. 1 b. 8 c. 6 d. 0 e. 13 f. 17 g. 19 h. –1

4. a. 2 b. 4 c. 1 d. 0 e. 5 f. 6 g. 9 h. 100

5. $5^{?} = 25$; since $5^2 = 25, \log_5 25 = 2$.

6. $2^{?} = 8$; since $2^3 = 8, \log_2 8 = 3$.

7. 1 **8.** 0

9. $100^{?} = 10$; since $100^{1/2} = \sqrt{100} = 10, \log_{100} 10 = \frac{1}{2}$.

10. –1

11. a. 2 b. 3 c. 2 d. 6 **12.** a. 5 b. 2 c. 1 d. 1

13. a. 0 b. 0 c. 0 d. 0 **14.** a. $\frac{1}{2}$ b. $\frac{1}{2}$ c. $\frac{1}{2}$ d. $\frac{1}{2}$

15. a. –1 b. –1 c. 0 d. –1 **16.** a. n b. 0 c. 1 d. 2

17. a. $\frac{1}{3}$ b. $\frac{1}{3}$ c. $\frac{1}{3}$ d. $\frac{1}{3}$

18. Let $y = \log_{64} 16 \Rightarrow 64^y = 16 \Rightarrow (4^3)^y = 4^2$

$$\Rightarrow 4^{3y} = 4^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

19. $-\frac{1}{2}$ **20**. $\frac{3}{4}$ **21**. $-\frac{2}{3}$ **22**. $\frac{4}{3}$ **23**. $-\frac{1}{5}$ **24**. 2

25. –3 **26**. 0 **27**. 5 **28**. $\frac{1}{2}$ **29**. $-\frac{3}{2}$

30. –2 **31**. $-\frac{1}{2}$ **32**. –3 **33**. –7 **34**. $\frac{1}{2}$

35. $\frac{2}{3}$ **36**. –2 **37**. $-\frac{1}{2}$ **38**. $-\frac{3}{2}$

39. pH = 11.9; it's a base (an alkali), since its pH is greater than 7.

40. $\text{pH} = -\log[\text{H}^+] = -\log(1.0\times10^{-7}) = -(-7) = 7$

41. a. 1.89 b. 5.55 c. 10.52
d. 11.04 e. 6.23 f. 0.10

42. $M = 4.6$

43. $M = 5.9$. Notice that this magnitude is only 1.3 Richter points higher than the previous answer, yet the earthquake was 100 times more powerful. Even a small difference in the Richter magnitude represents a huge difference in the actual power of the earthquake.

44. a. 6 b. 7 c. 1 d. 0 e. $\frac{1}{2}$ f. –1 g. –2 h. $-\frac{1}{2}$ i. 9

45. a. 1 b. 0 c. N

46. 10.47 (an alkali, or base)

47. 8.46

48. I can't just give the answer away.

"Learning is not the product of teaching.

Learning is the product of the activity of learners."

– John Holt, American psychologist and educator